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Vector measure range duality and factorizations of (D, p)-summing operators from Banach function spaces

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Abstract. We characterize the relationship between the space $L_1(\lambda')$ and the dual $L_1'(\lambda)$ of the space $L_1(\lambda)$, where (λ, λ') is a dual pair of vector measures with associated spaces of integrable functions $L_1(\lambda)$ and $L_1(\lambda')$ respectively. Since the result is rather restrictive, we introduce the notion of range duality in order to obtain factorizations of operators from Banach function spaces that are dominated by the integration map associated to the vector measure λ. We obtain in this way a generalization of the Grothendieck-Pietsch Theorem for p-summing operators.

Keywords: vector measure, integration, duality.

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Consider the space $L_1(\lambda)$ of (classes of) real functions that are integrable with respect to a vector measure λ, following the definition of Bartle, Dunford and Schwartz [1], and Lewis [10]. The aim of this paper is to study the properties and applications of the notion of dual pair of vector measures, and the relation with the corresponding spaces of integrable functions. Thus, Section 2 is devoted to the analysis of the natural relationship that appears between the notion of duality for a pair of vector measures (λ, λ') (see [8]) and the dual of the Banach space $L_1(\lambda)$. In this direction, we characterize when the dual space $L_1'(\lambda)$ can be represented in terms of $L_1(\lambda')$. The papers of Curbera [3] and Okada [13] are closely related to the general question of finding a good description for the

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dual space $L_1'(\lambda)$. However, as far as we know, no representation in terms of λ is known, but the results that can be obtained following the quoted scheme lead to the restrictive conclusion that the integration operator associated to the vector measure λ must be an isomorphism.

Therefore, we define in Section 3 the weaker notion of range duality between vector measures, and we prove that it can be successfully applied to generalize the Grothendieck-Pietsch Domination Theorem for operators from a Banach function space L that are dominated by the integration map of a vector measure. In order to do this, we introduce several compactness arguments for the topological space defined by the weak* closure of the unit ball of the space $L_1(\lambda')$ of a vector measure, that can be considered as a subspace of the dual of a projective tensor product of certain function spaces.

1 Preliminaries

Let (Ω, Σ) be a measurable space and X a (real) Banach space with dual X'. The closed unit ball of X will be denoted by B_X. Suppose that $\lambda : \Sigma \to X$ is a countably additive vector measure. We denote by $|\lambda|$ and $\|\lambda\|$ its variation and semivariation, respectively.

A measurable function $f : \Omega \to \mathbb{R}$ is integrable with respect to λ if for each $A \in \Sigma$ there is an element $\int_A f \, d\lambda \in X$ such that for every $x' \in X'$ the function f is $x'\lambda$-integrable and

$$\left\langle \int_A f \, d\lambda, x' \right\rangle = \int_A f \, d x'\lambda,$$

where $x'\lambda(B) := \langle \lambda(B), x' \rangle$ for every $B \in \Sigma$. The set of the (classes of $\|\lambda\|$-a.e. equal) functions endowed with the norm

$$\|f\|_\lambda = \sup \left\{ \int_\Omega |f| \, d \left| x'\lambda \right| \ : \ x' \in B_{X'} \right\}, \qquad f \in L_1(\lambda)$$

defines the Banach space $L_1(\lambda)$ (see [10, 11]). The (classes of) simple functions are dense in this space. If we consider the $\|\lambda\|$-almost everywhere order, $L_1(\lambda)$ becomes a Banach lattice. The norm

$$|\|f\||_\lambda = \sup_{A \in \Sigma} \left\| \int_A f \, d\lambda \right\|, \qquad f \in L_1(\lambda)$$

is equivalent to the above one. In fact $|\|f\||_\lambda \le \|f\|_\lambda \le 2|\|f\||_\lambda$ for every $f \in L_1(\lambda)$ (see [2]).

This space has been also studied in [9], [4, 3, 2], [13] and [18]. It is also known that $L_1(\lambda)$ can be described as a Banach (Köthe) function space over a positive measure which controls λ. This fact provides a useful characterization of the dual of $L_1(\lambda)$ (see [3, 13, 12]).

If λ is a bounded vector measure the integration operator can be defined as the linear and continuous map

$$I_\lambda : L_1(\lambda) \to X \mid I_\lambda(f) := \int_\Omega f \,\mathrm{d}\,\lambda, \qquad f \in L_1(\lambda).$$

The properties of the integration operator have been studied by Okada, Ricker and Rodríguez-Piazza in [14, 15, 16].

Let $\mathrm{rg}(\lambda)$ be the range of λ. It is said that λ is a *complete vector measure* if $\mathrm{span}\{\mathrm{rg}(\lambda)\}$ is dense in X, i.e. if the range of I_λ is dense in X.

Consider (λ, λ') a compatible pair of vector measures, that is, two vector measures $\lambda : \Sigma \to X$ and $\lambda' : \Sigma \to X'$. If (λ, λ') satisfies

(1) $\left\langle \lambda(A), \lambda'(B) \right\rangle = 0$ for disjoint $A, B \in \Sigma$,

(2) $\left\langle \lambda(A), \lambda'(A) \right\rangle \neq 0$ if at least one of the elements $\lambda(A)$ or $\lambda'(A)$ is non-zero,

then (λ, λ') is said to be a dual pair of vector measures. The duality relation between vector measures has been defined and studied by Kadets and Zheltukhin in [8].

In all the paper, we will suppose that the vector measures of a dual pair (or of a range dual pair, that will be defined in Section 3) (λ, λ') are countably additive. Our basic reference for vector measure theory is the book of Diestel and Uhl [6]. The notation for Banach spaces is standard. We will say that a subset B of a Banach space $(X, \|\cdot\|)$ is norming if $\sup_{x\in B} \left\langle x, x' \right\rangle$ defines a norm on X' that is equivalent to $\|\cdot\|'$. Throughout the paper, (Ω, Σ) and (Ω', Σ') will denote measurable spaces. The definitions and fundamental results on Banach (Köthe) function spaces and p-summing operators can be found in [12], [5] and [17].

2 Dual vector measures and function spaces duality

It is well known that $L_1(\lambda)$ is a Banach function space with weak unit [2]. Thus, if μ is a certain finite measure that controls λ (for instance, a Rybakov measure, see [6, 3]), then it is possible to characterize the dual space as a space of scalar functions by means of the duality relation

$$\langle f, g \rangle = \int_\Omega fg \,\mathrm{d}\,\mu,$$

where $f \in L_1(\lambda)$ and $g \in L_1'(\lambda)$ (see [12, Th. 1.b.14] and [3, 13]). For the representation of the dual space we use a positive measure related to the dual pair (λ, λ') that appears in a natural way. The *trace* of (λ, λ') (see [8, Sec. 3.1]) is the scalar measure $\mathrm{tr}(\lambda, \lambda') : \Sigma \to \mathbb{R}$ defined by

$$\mathrm{tr}(\lambda, \lambda')(A) := \left\langle \lambda(A), \lambda'(A) \right\rangle, \ A \in \Sigma.$$

Since λ is countably additive we obtain that $\mathrm{tr}(\lambda, \lambda')$ is also countably additive [8]. Consider the variation μ_t of $tr(\lambda, \lambda')$. It is clear that λ is absolutely continuous with respect to μ_t. Then the following lemmas clarify that the dual space $L_1'(\lambda)$ is also a function space over (Ω, Σ, μ_t) (see [4, 14]) and we can represent the duality by

$$\langle f, g \rangle = \int_\Omega fg \,\mathrm{d}\,\mu_t, \ f \in L_1(\lambda), \ g \in L_1'(\lambda).$$

Lemma 2.1. *Let (λ, λ') be a dual pair of vector measures, and let $\mu := \mathrm{tr}(\lambda, \lambda')$. For $f \in L_1(\lambda)$, $g \in L_1(\lambda')$ we have*

$$\left\langle \int_\Omega f \,\mathrm{d}\,\lambda, \int_\Omega g \,\mathrm{d}\,\lambda' \right\rangle = \int_\Omega fg \,\mathrm{d}\,\mu.$$

Proof. For $A, B \in \Sigma$ we have $\left\langle \lambda(A), \lambda'(B) \right\rangle = \mu(A \cap B)$ (see [8, Sec. 3.1]). The set of simple functions that are equal μ_t-a.e. is dense in the spaces $L_1(\lambda)$ and $L_1(\lambda')$. Thus, it will be enough to show the equality for simple functions. Take $f = \sum_{i=1}^n h_i \chi_{A_i}$ and $g = \sum_{i=1}^m k_i \chi_{B_i}$ where $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^m$ are sequences of disjoint subsets of Σ. We have

$$\begin{aligned}\left\langle \int_\Omega f \,\mathrm{d}\,\lambda, \int_\Omega g \,\mathrm{d}\,\lambda' \right\rangle &= \sum_{i=1}^n \sum_{j=1}^m h_i k_j \left\langle \lambda(A_i), \lambda'(B_j) \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m h_i k_j \mu(A_i \cap B_j) = \int_\Omega fg \,\mathrm{d}\,\mu.\end{aligned}$$

Moreover, for $A, B \in \Sigma$,

$$\left\langle \int_A f \,\mathrm{d}\,\lambda, \int_B g \,\mathrm{d}\,\lambda' \right\rangle \leq \left\| \int_A f \,\mathrm{d}\,\lambda \right\| \left\| \int_B g \,\mathrm{d}\,\lambda' \right\|,$$

and then

$$\left\langle \int_\Omega f \,\mathrm{d}\,\lambda, \int_\Omega g \,\mathrm{d}\,\lambda' \right\rangle \leq |\|f\||_\lambda |\|g\||_{\lambda'}.$$

A standard density argument completes the proof. □

Lemma 2.2. *Let* (λ, λ') *be a dual pair of vector measures such that* λ *is complete. Then the map*

$$i : L_1(\lambda') \to L_1'(\lambda) \mid i(f) = f$$

is a well defined continuous injection.

Proof. Let $\mu := \mathrm{tr}(\lambda, \lambda')$. By the Hahn decomposition there is a partition $\{A_+, A_-\}$ of Ω such that for $A \in \Sigma$ we have $\mu(A) = \mu(A \cap A_+) - \mu(A \cap A_-)$ (see [7, page 121]), and the measures $\mu(A \cap A_+)$ and $\mu(A \cap A_-)$ are finite and positive.

Consider the positive measure $\mu_t := \left|\mathrm{tr}(\lambda, \lambda')\right|$. It is clear that $\mu_t(A) = \mu(A \cap A_+) + \mu(A \cap A_-)$. Take $f \in L_1(\lambda')$ and $g \in L_1(\lambda)$. Lemma 2.1 gives

$$\left|\int_\Omega fg \,\mathrm{d}\,\mu_t\right| \leq \left|\int_{A_+} fg \,\mathrm{d}\,\mu\right| + \left|\int_{A_-} fg \,\mathrm{d}\,\mu\right| \leq 2\|\|f\|\|_{\lambda'}\|\|g\|\|_{\lambda}.$$

We therefore obtain that i is a continuous and well-defined map.

If $\int_\Omega fg \,\mathrm{d}\,\mu_t = 0$ for every $g \in L_1(\lambda)$, a direct calculation based on Lemma 2.1 shows that $\left\langle \int_\Omega g \,\mathrm{d}\,\lambda, \int_A f \,\mathrm{d}\,\lambda' \right\rangle = 0$ for all $g \in L_1(\lambda)$ and all $A \in \Sigma$. Since λ is a complete measure we obtain the conclusion. □

The statements (i) and (ii) of next proposition are proved in Theorem 3.6 of [8] for minimal measures.

Proposition 2.3. *Let* (λ, λ') *be a dual pair of complete vector measures. Then there exists a vector measure* λ^* *such that:*

(i) (λ, λ^*) *is a dual pair.*

(ii) *The measure* $\mathrm{tr}(\lambda, \lambda^*)$ *coincides with the variation of* $\mathrm{tr}(\lambda, \lambda')$. *In particular,* $\mathrm{tr}(\lambda, \lambda^*)$ *is a positive measure.*

(iii) $L_1(\lambda')$ *is isometric to* $L_1(\lambda^*)$.

Proof. Let $\mu := \mathrm{tr}(\lambda, \lambda')$ and $\mu_t := \left|\mathrm{tr}(\lambda, \lambda')\right|$. An application of the Radon-Nikodym Theorem gives a function $g \in L_1(\mu_t)$ such that $\mu(A) = \int_A g \,\mathrm{d}\,\mu_t$ (in fact, $|g| = 1$ μ_t-a.e.). We define the vector measure

$$\lambda^* : \Sigma \to X' \mid \lambda^*(A) := \int_A g^{-1} \,\mathrm{d}\,\lambda'.$$

The facts that $g^{-1} = g$ μ_t-a.e., μ_t controls λ, and $g \in L_1(\lambda')$ implies that the measure λ^* is well defined. For $A, B \in \Sigma$

$$\big\langle \lambda(A), \lambda^*(B) \big\rangle = \left\langle \lambda(A), \int_B g^{-1}\, \mathrm{d}\,\lambda' \right\rangle = \int_{A \cap B} g^{-1}\, \mathrm{d}\,\mu = \mu_t(A \cap B).$$

This and the fact that μ_t controls μ give that (λ, λ^*) is a dual pair of vector measures and its trace is μ_t. These implies (i) and (ii).

The multiplication operator

$$I_g : \big(L_1(\lambda'), |\|.\||_{\lambda'}\big) \to \big(L_1(\lambda^*), |\|.\||_{\lambda^*}\big) \mid I_g(f) := gf$$

is an isometry since

$$\big|\big\| I_g(f) \big\|\big|_{\lambda^*} = \sup_{A \in \Sigma} \left\| \int_A gf\, \mathrm{d}\,\lambda^* \right\| = \sup_{A \in \Sigma} \left\| \int_A gfg^{-1}\, \mathrm{d}\,\lambda' \right\| = |\|f\||_{\lambda'}.$$

This proves (iii). □

According to the above proposition we may suppose that the dual pair (λ, λ') is defined such that $\mathrm{tr}(\lambda, \lambda')$ is a positive measure. In this case we say that (λ, λ') is a *positive dual pair*.

Theorem 2.4. *Let (λ, λ') be a compatible pair of complete vector measures, $\lambda : \Sigma \to X$. Then the following are equivalent.*

(1) (λ, λ') *is a positive dual pair.*

(2) *The inclusion map $i : L_1(\lambda') \to L_1'(\lambda)$ is well-defined and factorizes as $i = I_\lambda' \circ I_{\lambda'}$, i.e. the diagram*

$$\begin{tikzcd} L_1(\lambda') \arrow[rr, "i"] \arrow[dr, "I_{\lambda'}"'] & & L_1'(\lambda) \\ & X' \arrow[ur, "I_\lambda'"'] & \end{tikzcd}$$

commutes, where I_λ' is the adjoint operator associated to I_λ.

Proof. (1) $\to$ (2). Set $\mu := \mathrm{tr}(\lambda, \lambda')$. Take $f \in L_1(\lambda')$. By Lemma 2.2 $f = i(f) \in L_1'(\lambda)$. Let $g \in L_1(\lambda)$. Since μ controls λ, we can write the duality between f and g as

$$\langle g, f \rangle = \langle g, i(f) \rangle = \int_\Omega gf\, \mathrm{d}\,\mu.$$

Now, Lemma 2.1 gives

$$\langle g, i(f)\rangle = \langle I_\lambda(g), I_{\lambda'}(f)\rangle = \left\langle g, I'_\lambda(I_{\lambda'}(f))\right\rangle.$$

This implies $i = I'_\lambda \circ I_{\lambda'}$.

(2) $\to$ (1). The definition of the duality in the function space $L_1(\lambda)$ gives a (finite) positive measure μ that controls λ. For $A, B \in \Sigma$, we have $\chi_A \in L_1(\lambda')$, $\chi_B \in L_1(\lambda)$ and

$$\begin{aligned}\mu(A \cap B) &= \int_\Omega \chi_A \chi_B \, d\mu = \langle \chi_B, i(\chi_A)\rangle \\ &= \langle I_\lambda(\chi_B), I_{\lambda'}(\chi_A)\rangle = \left\langle \lambda(B), \lambda'(A)\right\rangle.\end{aligned}$$

This equality holds for each pair of subsets $A, B \in \Sigma$. Direct calculations give conditions (1) and (2) of the definition of dual pair for (λ, λ'). Since μ is positive we have that (λ, λ') is a positive dual pair. $\square$

Next we will prove the main result of this section. We show that the coincidence between the duality for the vector measures and the duality between $L_1(\lambda)$ and $L_1(\lambda')$ is only satisfied when the integration map is an isomorphism. We begin with an example.

Example 2.5. Let $1 < p < \infty$ and consider the usual space $L_p([0,1], \Sigma_0, \mu_0)$ of p-integrable functions on $[0,1]$. We define the vector measure $\lambda_p : \Sigma_0 \to L_p[0,1]$ by mean of $\lambda_p(A) := \chi_A$. It is a countably additive (hence bounded) vector measure. Moreover, if p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$, then $(\lambda_p, \lambda_{p'})$ is a dual pair of vector measures.

In this case, the integration map $I_p : L_p[0,1] \to L_p[0,1]$ defined as $I_p(f) := \int_\Omega f \, d\lambda_p$ is an isomorphism, since for every simple function $f = \sum_{i=1}^n h_i \chi_{A_i}$ (where $\{A_i\}_{i=1}^n$ are disjoint subsets of Σ_0),

$$I_p(f) = \int_\Omega f \, d\lambda_p = \sum_{i=1}^n h_i \lambda_p(A_i) = \sum_{i=1}^n h_i \chi_{A_i},$$

and

$$\left|\left\| I_p(f)\right\|\right|_{\lambda_p} = \sup_{A \in \Sigma_0} \left\| \int_A f \, d\lambda_p \right\|_{L_p} = \sup_{A \in \Sigma_0} \left\{ \int_A |f|^p \, d\mu_0 \right\}^{\frac{1}{p}} = \|f\|_{L_p}.$$

It is clear that we can represent the dual of $L_1(\lambda_p)$ as $L_1(\lambda_{p'})$, i.e. the dual pair $(\lambda_p, \lambda_{p'})$ of vector measures leads to a direct representation of the duality of the corresponding function spaces. On the other hand, the integration maps I_p and $I_{p'}$ define isomorphic relations between the function spaces and the Banach spaces where the vector measures take their values.

The following theorem shows that this situation holds for the general case, that is, function spaces related to the measures of a dual pair are dual spaces if and only if the corresponding integration maps are isomorphisms.

Theorem 2.6. *Let (λ, λ') be a dual pair of complete vector measures. Then the following are equivalent.*

(1) *$L_1(\lambda')$ is a subspace of $L_1'(\lambda)$.*

(2) *The quotient map $\overline{I}_\lambda : \dfrac{L_1(\lambda)}{\ker I_\lambda} \to X$ is an isomorphism.*

(3) *$I_{\lambda'}$ defines an isomorphism between $L_1(\lambda')$ and X'.*

(4) *I_λ defines an isomorphism between $L_1(\lambda)$ and X.*

Proof. As a consequence of Proposition 2.3, we can suppose that (λ, λ') is a positive dual pair. Then we can define the dual space $L_1'(\lambda)$ by the duality $\int_\Omega fg \, \mathrm{d}\,\mu$, with the measure $\mu = \mathrm{tr}(\lambda, \lambda')$.

(2) $\to$ (1). Lemma 2.2 gives the continuity of the map $i : L_1(\lambda') \to L_1'(\lambda)$. Thus we just need to prove that there is a constant $Q > 0$ such that

$$\|f\|_{L_1(\lambda')} \leq Q \|f\|_{L_1'(\lambda)}$$

for each $f \in L_1'(\lambda)$. First, we know that

$$|\|f\||_{\lambda'} = \sup_{A \in \Sigma} \left\| \int_A f \, \mathrm{d}\,\lambda' \right\| = \sup \left\{ \left\langle x, \int_A f \, \mathrm{d}\,\lambda' \right\rangle \ : \ A \in \Sigma, \ x \in B_X \right\}.$$

Since $\overline{I}_\lambda$ is an isomorphism, the definition of the norm on the quotient space $\frac{L_1(\lambda)}{\ker I_\lambda}$ makes clear that there is a function $g \in B_{L_1(\lambda)}$, a set $A \in \Sigma$ and a constant $Q > 0$ (which does not depend on g or A) such that

$$|\|f\||_{\lambda'} \leq Q \left\langle \int_\Omega g \, \mathrm{d}\,\lambda, \int_A f \, \mathrm{d}\,\lambda' \right\rangle.$$

Then, by Lemma 2.1

$$|\|f\||_{\lambda'} \leq Q \int_A gf \, \mathrm{d}\,\mu = Q \int_\Omega (g\chi_A) f \, \mathrm{d}\,\mu.$$

Taking into account that $|\|g\chi_A\||_\lambda \leq |\|g\||_\lambda \leq 1$ we obtain

$$|\|f\||_{\lambda'} \leq Q\|f\|_{L_1'(\lambda)}.$$

Hence $L_1(\lambda')$ can be identified with a subspace of $L_1'(\lambda)$.

Now let us show that (1) $\to$ (3). Theorem 2.4 gives the factorization through X'. Since the identity map i is an injection as a consequence of (1), $I_{\lambda'}$ is also injective. There is a $K > 0$ such that, for every $f \in L_1(\lambda')$,

$$\begin{aligned}\|f\|_{L_1(\lambda')} &\leq K \sup_{|\|g\||_\lambda \leq 1} \int_\Omega fg \,\mathrm{d}\,\mu \\ &= K \sup_{|\|g\||_\lambda \leq 1} \left\langle \int_\Omega g \,\mathrm{d}\,\lambda, \int_\Omega f \,\mathrm{d}\,\lambda' \right\rangle \leq K \left\| \int_\Omega f \,\mathrm{d}\,\lambda' \right\| = K\|I_{\lambda'}(f)\|.\end{aligned}$$

These inequalities and the completeness of the measure λ' give the result.

Finally we show (3) $\to$ (4). There are constants K and Q such that

$$\begin{aligned}|\|f\||_\lambda &= \sup_{A\in\Sigma} \left\| \int_A f \,\mathrm{d}\,\lambda \right\| = \sup \left\{ \left\langle \int_A f \,\mathrm{d}\,\lambda, x' \right\rangle \ : \ \|x'\| \leq 1,\ A \in \Sigma \right\} \\ &\leq K \sup \left\{ \left\langle \int_A f \,\mathrm{d}\,\lambda, \int_\Omega g \,\mathrm{d}\,\lambda' \right\rangle \ : \ \|g\|_{\lambda'} \leq 1,\ A \in \Sigma \right\} = \ (*)\end{aligned}$$

Since for every $A \in \Sigma$ and $g \in B_{L_1(\lambda')}$ we have $g\chi_A \in L_1(\lambda')$, $\|g\chi_A\|_{\lambda'} \leq \|g\|_{\lambda'} \leq 1$ and $\left\langle \int_A f \,\mathrm{d}\,\lambda, \int_\Omega g \,\mathrm{d}\,\lambda' \right\rangle = \int_A fg \,\mathrm{d}\,\mu = \left\langle \int_\Omega f \,\mathrm{d}\,\lambda, \int_\Omega g\chi_A \,\mathrm{d}\,\lambda' \right\rangle$, we can write

$$(*) \ = K \sup \left\{ \left\langle \int_\Omega f \,\mathrm{d}\,\lambda, \int_\Omega g \,\mathrm{d}\,\lambda' \right\rangle \ : \ \|g\|_{\lambda'} \leq 1 \right\} \leq KQ\|I_\lambda(f)\|.$$

Thus, the completeness of λ implies that I_λ is an isomorphism.

Since the fact that (4) implies (2) is obvious, this finishes the proof. □

Corollary 2.7. *Let (λ, λ') be a dual pair of complete vector measures. Suppose that the set of (classes of) simple functions is dense in $L_1'(\lambda)$. Then the following are equivalent.*

(1) $L_1(\lambda') = L_1'(\lambda)$.

(2) $I_{\lambda'}$ *defines an isomorphism between* $L_1(\lambda')$ *and* X'.

(3) I_λ *defines an isomorphism between* $L_1(\lambda)$ *and* X.

3 The range dual of a space $L_1(\lambda)$

After the results of Section 2, we know that the only case that we can represent the dual of $L_1(\lambda)$ as a space $L_1(\lambda')$ is the trivial case when it is isomorphic to the space X where the vector measure is defined. This leads to the definition of a weak duality relation between $L_1(\lambda)$ and other space $L_1(\gamma)$ when γ is a (countably additive) vector measure $\gamma : \Sigma' \to X'$, that is given by the bilinear form

$$(f, g) := \left\langle \int_\Omega f \, d\lambda, \int_{\Omega'} g \, d\lambda' \right\rangle, \qquad f \in L_1(\lambda), g \in L_1(\gamma).$$

The inequality

$$(f, g) \leq |\|f\||_\lambda |\|g\||_\gamma, \qquad f \in L_1(\lambda), g \in L_1(\gamma)$$

proves the continuity of this bilinear map.

Definition 3.1. *Consider the space $L_1(\lambda)$ of a countably additive vector measure $\lambda : \Sigma \to X$. We define the range dual of $L_1(\lambda)$ as the linear space given by the range of the adjoint operator I'_λ, i.e.*

$$(L_1(\lambda))^R = \{\phi_{x'} \in (L_1(\lambda))' : \phi_{x'}(f) := \left\langle \int_\Omega f \, d\lambda, x' \right\rangle, \quad \textit{for } x' \in X', \; f \in L_1(\lambda)\},$$

endowed with the norm $\|\phi_{x'}\|^R_\lambda = \|\phi_{x'}\|_{L'_1(\lambda)}$.

The linear map $R : X' \to (L_1(\lambda))^R$ given by $R(x') = \phi_{x'}$, $x' \in X'$, is continuous, since

$$|\phi_{x'}(f)| \leq \|f\|_\lambda \|x'\|.$$

The properties of a range dual of a space $L_1(\lambda)$ are obviously associated to the relationship between this function space and the Banach space X where the vector measure is defined. In a certain sense, it represents the elements of the dual space $L_1(\lambda)$ that can be defined by mean of the elements of X'. The following proposition clarifies this relation. It is a direct consequence of the properties of the adjoint operators.

Proposition 3.2. *Consider the space $L_1(\lambda)$ of a countably additive vector measure λ. If I_λ is open, or onto, or $I_\lambda(B_{L_1(\lambda)})$ is norming, then R defines an isomorphism between $(L_1(\lambda))^R$ and X'.*

Moreover, if R defines such an isomorphism, then $I_\lambda(B_{L_1(\lambda)})$ is norming and $(L_1(\lambda))^R$ (and then X') is isomorphic to a closed subspace of $(L_1(\lambda))'$.

Definition 3.3. *Let $\lambda : \Omega \to X$ and $\lambda' : \Omega' \to X'$ be countably additive vector measures. We say that the couple (λ, λ') is a range dual pair if*

(1) *the seminorm*

$$\|g\|_{(\lambda,\lambda')} = \sup_{f \in B_{L_1(\lambda)}} \left\langle \int_\Omega f \, \mathrm{d}\,\lambda, \int_{\Omega'} g \, \mathrm{d}\,\lambda' \right\rangle$$

is equivalent to $\| \int_{\Omega'} g \, \mathrm{d}\,\lambda' \|$ for every $g \in L_1(\lambda')$, and

(2) *the seminorm*

$$\|f\|_{(\lambda',\lambda)} = \sup_{g \in B_{L_1(\lambda')}} \left\langle \int_\Omega f \, \mathrm{d}\,\lambda, \int_{\Omega'} g \, \mathrm{d}\,\lambda' \right\rangle$$

is equivalent to $\| \int_\Omega f \, \mathrm{d}\,\lambda \|$ for every $f \in L_1(\lambda)$.

A direct consequence of Proposition 3.2 is the following corollary, that clearly shows that uniqueness is not a defining property of the range dual pair relation between vector measures.

Corollary 3.4. *If I_λ and $I_{\lambda'}$ are open, or onto, or $I_\lambda(B_{L_1(\lambda)})$ and $I_{\lambda'}(B_{L_1(\lambda)})$ are norming, then (λ, λ') is a range dual pair.*

Definition 3.5. *Let $\lambda : \Omega \to X$ and $\lambda' : \Omega' \to X'$ be countably additive vector measures. We define the seminorms*

$$\|f\|_\lambda^{\lambda'} := \sup_{A \in \Sigma} \| \int_A f \, \mathrm{d}\,\lambda \|_{(\lambda',\lambda)}, \qquad f \in L_1(\lambda),$$

and

$$\|g\|_{\lambda'}^{\lambda} := \sup_{A \in \Sigma'} \| \int_A g \, \mathrm{d}\,\lambda' \|_{(\lambda,\lambda')}, \qquad g \in L_1(\lambda').$$

Remark 3.6. Consider a range dual pair (λ, λ'). It is clear by construction that

(1) $\| \cdot \|_\lambda^{\lambda'}$ and $\| \cdot \|_\lambda$ are equivalent on $L_1(\lambda)$, and

(2) $\| \cdot \|_{\lambda'}^{\lambda}$ and $\| \cdot \|_{\lambda'}$ are equivalent on $L_1(\lambda')$.

Therefore, range dual pairs provide a right framework to define the topology of the spaces of integrable functions by mean of a weak duality relation based on the computation of the corresponding integrals.

Lemma 3.7. *If λ and λ' define a range dual pair and μ is a Rybakov measure for λ', then $L_1(\lambda')$ is isomorphic to a subspace of*

$$(L_\infty(\mu) \otimes_\pi L_1(\lambda))'.$$

Proof. Let $g \in L_1(\lambda')$. Let us define the function $\Phi_g : L_\infty(\mu) \otimes_\pi L_1(\lambda) \to R$ by

$$\Phi_g(h \otimes f) := \left\langle \int_\Omega f \, \mathrm{d}\,\lambda, \int_{\Omega'} hg \, \mathrm{d}\,\lambda' \right\rangle, \qquad h \otimes f \in L_\infty(\mu) \otimes_\pi L_1(\lambda),$$

and extended by linearity to the whole tensor product. It is well defined, since it is clear that $\Phi_g(z)$ does not depend on the particular representation of the tensor z that is considered. The norm $\|g\|_{\lambda'}$ can also be computed as the supremum $\sup_{B_{L_\infty(\mu)}} \| \int_{\Omega'} hg \, \mathrm{d}\,\lambda' \|$ (see [19]). Thus, for every

$$z = \sum_{i=1}^n h_i \otimes f_i \in L_\infty(\mu) \otimes L_1(\lambda)$$

we have

$$\begin{aligned} \Phi_g\left(\sum_{i=1}^n h_i \otimes f_i \right) &\leq \sum_{i=1}^n \| \int_\Omega f_i \, \mathrm{d}\,\lambda \| \| \int_{\Omega'} h_i g \, \mathrm{d}\,\lambda' \| \| \\ &\leq \|g\|'_\lambda \left(\sum_{i=1}^n \|h_i\|_{L_\infty(\mu)} \|f_i\|_\lambda \right), \end{aligned}$$

and then

$$\Phi_g(z) \leq \|g\|_{\lambda'} \pi(z).$$

Moreover, since λ and λ' define a range dual pair, we have that the norm

$$\sup_{A \in \Sigma', f \in B_{L_1(\lambda)}} \Phi_g(\chi_A \otimes f)$$

is equivalent to $\| \cdot \|_\lambda$, and obviously $\chi_A \in L_\infty(\mu)$ for every $A \in \Sigma'$. This proves the result. $\square$

Now, let us consider the weak* topology on $(L_\infty(\mu) \otimes_\pi L_1(\lambda))'$. By Lemma 3.7, $B_{L_1(\lambda')}$ can be considered as a norm closed subset on this space. Since it is convex, it is closed also for the weak topology, but it is not in general closed for the inherited weak* topology. Let us denote by $\overline{B}_{L_1(\lambda')}$ the weak* closure of this set. The following lemma shows that we can still consider the elements of $\overline{B}_{L_1(\lambda')}$ (in a weak sense) as elements of a function space.

Lemma 3.8. *Let $f \in L_1(\lambda)$ and $\phi \in \overline{B}_{L_1(\lambda')}$. Then there is a function $\phi_f \in L_1(\mu)$ such that for every $h \in L_\infty(\mu)$,*

$$\langle h \otimes f, \phi_f \rangle = \int_{\Omega'} h\phi_f \, d\mu.$$

Proof. Since $\phi \in \overline{B}_{L_1(\lambda')}$, there is a net $(g_\tau)_{\tau \in T} \subset B_{L_1(\lambda')}$ so that $\lim_{\tau \in T} g_\tau = \phi$. Let $f \in L_1(\lambda)$, and consider the measures $\nu_{f,\tau}$ given by

$$\nu_{f,\tau}(A) := \left\langle \int_\Omega f \, d\lambda, \int_{\Omega'} \chi_A g_\tau \, d\lambda' \right\rangle, \qquad A \in \Sigma',$$

and the set function

$$\nu_f(A) := \lim_{\tau \in T} \nu_{f,\tau}(A) = \phi(\chi_A \otimes f) \qquad A \in \Sigma'.$$

Note that $|\nu_f(A)| \le \|f\|_\lambda$ for every $A \in \Sigma'$ and then ν_f is a finite measure. It is absolutely continuous with respect to μ, since so is each measure $\nu_{f,\tau}$. Thus, there is a function $\phi_f \in L_1(\mu)$ such that $\nu_f(A) = \int_A \phi_f \, d\mu$ for every $A \in \Sigma'$. Therefore, for every $h \in L_\infty(\mu)$ we obtain

$$\phi(h \otimes f) = \int_{\Omega'} h\phi_f \, d\mu. \qquad \square$$

Lemma 3.8 allows us to introduce the following notation for the extension of the action of the elements of $B_{L_1(\lambda')}$ on $L_\infty(\mu) \otimes L_1(\lambda)$. If $\phi \in \overline{B}_{L_1(\lambda')}$, $h \in L_\infty(\mu)$ and $f \in L_1(\lambda)$, we define

$$(f, h\phi) := \int_{\Omega'} h\phi_f \, d\mu.$$

If $h = \chi_{\Omega'}$ we simply write (f, ϕ). Note that for every element $g \in B_{L_1(\lambda')}$,

$$(f, hg) = \left\langle \int_\Omega f \, d\lambda, \int_{\Omega'} hg \, d\lambda' \right\rangle.$$

Definition 3.9. *We say that a Banach function space L is compatible with a vector measure λ if the identity map $Id : L \to L_1(\lambda)$ given by $Id(f) := f$, $f \in L$, is well defined, continuous and its range is dense in $L_1(\lambda)$.*

Definition 3.10. *Consider a Banach function space L that is compatible with the countably additive vector measure λ of the range dual pair $D = (\lambda, \lambda')$. Let $1 \leq p < \infty$. Let Y be a Banach space, and consider an operator $T : L \to Y$. We say that T is (D, p)-summing if there is a positive constant K such that for every finite set of functions $f_1, \ldots, f_n \in L$, the inequality*

$$\sum_{i=1}^{n} \|T(f_i)\|^p \leq K^p \sup_{g \in B_{L_1(\lambda')}} \sum_{i=1}^{n} \left| \left\langle \int_\Omega f_i \, d\lambda, \int_{\Omega'} g \, d\lambda' \right\rangle \right|^p$$

holds. We denote by $\pi_{(D,p)}$ to the infimum of all such constants K.

Theorem 3.11. *Let L be a Banach function space that is compatible with the countably additive vector measure λ of the range dual pair $D = (\lambda, \lambda')$. Let Y be a Banach space, $1 \leq p < \infty$, and consider an operator $T : L \to Y$. Then the following statements are equivalent.*

(1) *T is (D, p)-summing.*

(2) *There is a positive constant K and a regular Borel probability measure η on the compact set $\overline{B}_{L_1(\lambda')}$ such that*

$$\|T(f)\| \leq K \left(\int_{\overline{B}_{L_1(\lambda')}} |(f, \phi)|^p \, d\eta(\phi) \right)^{\frac{1}{p}}$$

for every $f \in L$.

Moreover, the infimum of such constants K equals $\pi_{(D,p)}$.

Proof. It follows the lines of the proof of the Grothendieck-Pietsch Domination Theorem (see 2.12 in [5]). To see (2) $\to$ (1), consider a finite family of functions $f_1, \ldots, f_n \in L$. Then there is a constant $K > 0$ and a probability measure η such that

$$\sum_{i=1}^{n} \|T(f_i)\|^p \leq K^p \sum_{i=1}^{n} (\int_{\overline{B}_{L_1(\lambda')}} |(f_i, \phi)|^p \, d\eta(\phi)) = \qquad (*)$$

Thus, since $B_{L_1(\lambda')}$ is weak* dense in $\overline{B}_{L_1(\lambda')}$ and the functions $|(f_i, \phi)|$ are continuous for the weak* topology, we obtain

$$(*) \leq K^p \sup_{\phi \in \overline{B}_{L_1(\lambda')}} \sum_{i=1}^{n} |(f_i, \phi)|^p \leq K^p \sup_{g \in B_{L_1(\lambda')}} \sum_{i=1}^{n} \left| \left\langle \int_\Omega f_i \, d\lambda, \int_{\Omega'} g \, d\lambda' \right\rangle \right|^p$$

This gives (1). For the converse, consider any finite set of functions $\mathcal{F} = \{f_i \in L_1(\lambda) : i = 1, ..., n\}$. We can define the function

$$\psi_{\mathcal{F}}(g) = \sum_{i=1}^{n} \|T(f_i)\|^p - \pi^p_{(D,p)} \left| \left\langle \int_\Omega f_i \, d\,\lambda, \int_{\Omega'} g \, d\,\lambda' \right\rangle \right|^p.$$

It is a continuous function on $B_{L_1(\lambda')}$ with respect to the weak* topology, which definition can be extended by continuity to the whole $\overline{B}_{L_1(\lambda')}$. Let M be the set of all such functions $\mathcal{F}$. It can be easily shown that it is a convex set. If we denote by C the positive cone of the space of continuous functions $C(\overline{B}_{L_1(\lambda')})$, i.e.

$$C = \{\gamma \in C(\overline{B}_{L_1(\lambda')}) | \gamma(\phi) > 0, \quad \text{for every} \quad \phi \in \overline{B}_{L_1(\lambda')}\}.$$

Since T is (D, p)-summing it is clear that C and M are disjoint. Since C is open and convex, the Hahn-Banach Theorem gives an element $\eta \in (C(\overline{B}_{L_1(\lambda')}))'$ and a constant k such that

$$\langle \psi, \eta \rangle \leq k < \langle \gamma, \eta \rangle$$

for every $\psi \in M$, $\gamma \in C$. Since $0 \in M$ and every positive constant function h belongs to C, it follows that $k = 0$. Thus, η is a positive regular Borel measure, that we can suppose that is a probability measure. Therefore,

$$\int_{\overline{B}_{L_1(\lambda')}} \psi \, d\,\eta \leq 0$$

for every $\psi \in M$. This gives the result. □

The definition of (D, p)-summing operators directly provides the following factorization theorem. The reader can find a study of related factorizations in [18].

Lemma 3.12. *In the conditions of Definition 3.10, if T is (D, p)-summing then we can factorize it as $T = T_0 \circ Id$, where $I_d : L \to L_1(\lambda)$ is the corresponding identity map and $T_0 : L_1(\lambda) \to Y$ is defined by $T_0(f) = T(f)$ for every $f \in L$, and by continuity when f does not belong to L. Moreover, T_0 is also (D, p)-summing.*

Proof. Since Id is continuous, it is enough to prove the continuity of T_0. But for every $f \in L$,

$$\|T_0(f)\| \leq \pi_{(D,p)} \sup_{g \in B_{L_1(\lambda')}} \left| \left\langle \int_\Omega f \, d\,\lambda, \int_{\Omega'} g \, d\,\lambda' \right\rangle \right| \leq \pi_{(D,p)} \|f\|_\lambda.$$

Thus, the continuity of T_0 holds. Consider a finite family of functions $f_i \in L_1(\lambda)$, $i = 1, ..., n$. Let $\epsilon > 0$. The density of $Id(L)$ and the continuity of the expression

$$\sup_{g \in B_{L_1(\lambda')}} \sum_{i=1}^{n} | \left\langle \int_{\Omega} f_i \, \mathrm{d}\,\lambda, \int_{\Omega'} g \, \mathrm{d}\,\lambda' \right\rangle |^p$$

with respect to the norm topology of $L_1(\lambda)$ gives that there are $f_i' \in L$ such that

$$\|T_0(f_i) - T_0(f_i')\| < \frac{\epsilon}{n}, \qquad i = 1, ..., n.$$

and

$$\sup_{g \in B_{L_1(\lambda')}} \left(\sum_{i=1}^{n} | \left\langle \int_{\Omega} f_i - f_i' \, \mathrm{d}\,\lambda, \int_{\Omega'} g \, \mathrm{d}\,\lambda' \right\rangle |^p\right)^{\frac{1}{p}} < \epsilon.$$

This can be found for every $\epsilon > 0$. Thus, a direct calculation using the inequality that holds for $\{f_i' : i = 1, ..., n\}$ by the fact that T is (D, p)-summing gives the result. □

In particular, each (D, p)-summing operator T is dominated by the integration operator I_λ, i.e. for every $f \in L$,

$$\|T(f)\| \leq \pi_{(D,p)} \| \int_{\Omega} f \, \mathrm{d}\,\lambda \|.$$

Although this property gives a strong restriction when λ is a scalar measure, this is not the case in the vectorial situation (see Remark 3.14).

Theorem 3.13. *In the conditions of Definition 3.10, an operator $T : L \to Y$ is (D, p)-summing if and only if there is a factorization as follows,*

$$\begin{tikzcd}
L \arrow[rr, "T"] \arrow[d, "Id"'] & & Y \\
L_1(\lambda) \arrow[d, "J"'] & & \\
G \subset C(\overline{B}_{L_1(\lambda')}) \arrow[rr, "I_p"] & & I_p(G) \subset L_p(\overline{B}_{L_1(\lambda')}, \eta) \arrow[uu, "S"']
\end{tikzcd}$$

where

(a) *J is the map given by $J(f)(g) := \left\langle \int_\Omega f \, \mathrm{d}\,\lambda, \int_{\Omega'} g \, \mathrm{d}\,\lambda' \right\rangle$, for every $f \in L_1(\lambda)$ and $g \in L_1(\lambda')$, and G is the closure of $J(L_1(\lambda))$.*

(b) I_p *is the identity continuous map defined as* $I_p(h) = h$, $h \in C(\overline{B}_{L_1(\lambda')})$.

(c) S *is the map defined from the closure of* $I_p(G)$ *by continuity by the expression* $S(I_p(J(Id(f)))) = T(f)$, $f \in L$.

Moreover, $\|S\| = \pi_{(D,p)}$.

Proof. We first show that every (D, p)-summing operator T factorizes in this way. Lemma 3.12 gives a previous factorization; there is a (D, p)-summing operator $T_0 : L_1(\lambda) \to Y$ such that $T = T_0 \circ Id$. So we only need to obtain the factorization scheme for T_0. The function $J : L_1(\lambda) \to C(\overline{B}_{L_1(\lambda')})$ is continuous, since obviously the unit ball of $L_1(\lambda')$ is dense in its closure with respect to the weak* topology, and then

$$\|J(f)\| = \sup_{g \in B_{L_1(\lambda')}} \left\langle \int_\Omega f \, d\lambda, \int_{\Omega'} g \, d\lambda' \right\rangle \leq \|f\|_{L_1(\lambda)}, \qquad f \in L_1(\lambda).$$

The operator I_p is well defined and continuous, and then we can consider the closure $\overline{I_p(G)}$ of $I_p(G)$.

Let w be a function of $\overline{I_p(G)}$ and let $\epsilon > 0$. Then there is a function $f \in L$ such that $I_p(J(Id(f))) = y$ and $\|y - w\| < \epsilon$. It satisfies $S(y) = T(f)$ and

$$\|S(y)\| \leq \pi_{(D,p)} \left(\int_{\overline{B}_{L_1(\lambda')}} |(f, \phi)|^p \, d\eta(\phi) \right)^{\frac{1}{p}} = \pi_{(D,p)} \|y\|_{L_p(\overline{B}_{L_1(\lambda')}, \eta)},$$

as a consequence of Theorem 3.11. Thus, the same inequality holds for w and then S is continuous. This proves the factorization for (D, p)-summing maps.

For the converse, it is enough to use the continuity of S in the same way that has been used above. For every $f \in L$, $T(f) = S(I_p(J(Id(f))))) = S((f, \cdot))$, and

$$\|T(f)\| = \|S((f, \cdot))\| \leq \|S\| \left(\int_{\overline{B}_{L_1(\lambda')}} |(f, \phi)|^p \, d\eta(\phi) \right)^{\frac{1}{p}}.$$

Therefore, T is (D, p)-summing. These arguments also show that the norm of S equals $\pi_{(D,p)}$. □

Remark 3.14. Note that after the results of Section 2 and [4], we can always find a dual pair of vector measures that defines the duality between L and L', *via* the representation that can be obtained for every Banach function space with weak unit by mean of the countably additive vector measure $\lambda : \Sigma \to L$ defined as $\lambda(A) = \chi_A$. Thus, it can be easily shown that in this case Theorem 3.13 gives the classical factorization theorem for p-summing operators. In this sense, we have obtained a generalization of the Grothendieck-Pietsch Theorem for Banach function spaces.

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